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The Existence and Uniqueness of the Solution for Stochastic Functional Differential Equations with Infinite Delay at Phase Space B*

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Abstract: The existence and uniqueness of the solutions to stochastic functional differential equations with infinite delay at phase space B is considered in this paper. Under the weaken linear growth condition and uniform Lipschitz condition, the system has a unique solution on the interval $[0,\infty)$. Then the moment estimate for the error between approximate solution and accurate solution is given.

Keywords: stochastic functional differential equations; existence; uniqueness; infinite delay

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1 Introduction

Mao[1] investigated the stochastic functional differential equations

$$\mathrm{d}X(t)=f(X_t,t)\,\mathrm{d}t+g(X_t,t)\,\mathrm{d}B(t),\quad t_0\le t\le T,\tag{1}$$

at phase space $C([-\tau,0],R^d)$, where $f:R^d\times[t_0,T]\to R^d$, $g:R^d\times[t_0,T]\to R^{d\times m}$ are Borel measurable functions. The initial value is given by $X_{t_0}=X_0\in\mathcal{M}^2((t_0-\tau,t_0];R^d)$. Under the uniform Lipschitz condition and linear growth condition, he obtained that the systems (1) had a unique solution $X(t)\in\mathcal{M}^2([t_0-\tau,T];R^d)$, where $t_0\in R$, $T>0$, $\tau>0$.

Recently, Wei[2] and Xu[3] had proved the existence and uniqueness of solutions for stochastic functional differential equations with infinite delay at phase space BC and B, respectively. For simplicity, throughout this paper, we take $t_0=0$. The authors improve linear growth condition of [3] to weaken linear growth condition in this paper, consider a d-dimensional stochastic functional differential equations with infinite delay at phase space B

$$\mathrm{d}X(t)=f(X_t,t)\,\mathrm{d}t+g(X_t,t)\,\mathrm{d}B(t),\quad 0\le t\le T,\tag{2}$$

where $X_t=\{X(t+\theta):-\infty<\theta\le 0\}$ can be regarded as a B-value stochastic process, $f:B\times[0,T]\to R^d$ and $g:B\times[0,T]\to R^{d\times m}$ are Borel measurable functions. The initial

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value is given by

$$X_0 = \xi = \{\xi(\theta) : -\infty < \theta \le 0\} \in \mathcal{M}^2((-\infty, 0]; R^d). \tag{3}$$

Under the weaken linear growth condition and uniform Lipschitz condition, the result that the system (2) has a unique solution on the interval $[0, \infty)$ is obtained. Further, the moment estimate for the error between approximate solution and accurate solution is given.

2 Preliminary

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions, and $B(t)$ be an m-dimensional Brownian motion defined on complete probability space, that is $B(t) = (B_1(t), B_2(t), \cdots, B_m(t))^T$. Let B be the real vector space of real functions mapping from $(-\infty, 0]$ to R^n, and denote the semi-norm by $|\cdot|_B$ (see [4,5] for details). Phase space B satisfies the following conditions:

(B_1) For any $\varphi \in B$ and any $0 < a \le \infty$, if x is a R^n-value function defined on $(-\infty, \sigma + a)$ with $x_\sigma = \varphi$, and x is a continuous function on $[\sigma, \sigma + a)$. Then for any $t \in [\sigma, \sigma + a)$, it follows that $x_t \in B$, moreover x_t is continuous with respect to t.

(B_2) There exists a continuous function $K(\beta) > 0$ such that $|\varphi|_B \le K(\beta)|\varphi|^{[-\beta,0]} + |\varphi|_\beta$ for any $\varphi \in B$ and any $\beta \in [0, \infty)$, where

$$|\varphi|^{[-\beta,0]} = \sup\{\, |\varphi(\theta)|, \ -\beta \le \theta \le 0\}.$$

(B_3) There exists a continuous function $M(\beta) > 0$ such that $|\tau^\beta \varphi|_\beta \le M(\beta)|\varphi|_B$ for any $\varphi \in B$ and for $\beta \in [0, \infty)$, the linear operator $\tau^\beta : B \to B^\beta$ is defined by $[\tau^\beta \varphi](\theta) = \varphi(\beta + \theta)$.

(B_4) There exists a positive number N such that $|\varphi(0)| \le N|\varphi|_B$ for any $\varphi \in B$.

Hale and Kato[4] had proved that phase space B was a Banach space.

Definition 2.1 The R^d-value stochastic process $X(t)$ defined on $-\infty < t \le T$ is called the solution of the system (2) with initial data (3), if it has the following properties:

(i) $X(t)$ is continuous and $\{X(t)\}_{0\le t\le T}$ is $\mathcal{F}_t$-adapted;

(ii) $\{f(X_t, t)\} \in \mathcal{L}^1([0, T]; R^d)$ and $\{g(X_t, t)\} \in \mathcal{L}^2([0, T]; R^{d\times m})$;

(iii) $X_0 = \xi$, and for each

$$0 \le t \le T, \quad X(t) = \xi(0) + \int_0^t f(X_s, s)\mathrm{d}s + \int_0^t g(X_s, s)\mathrm{d}B(s), \quad \text{a.s.}$$

$X(t)$ is said to be unique, if any other solution $\bar{X}(t)$ is distinguishable from $X(t)$, that is

$$P\{\, X(t) = \bar{X}(t), \ \text{for all} \ -\infty < t \le T\} = 1.$$

3 Existence and uniqueness of the solutions

Theorem 3.1 Assume that there exists two positive constants K and $\bar{K}$ such that (weaken linear growth condition) for any $t \in [0, T]$

$$|f(0, t)|^2 \vee |g(0, t)|^2 \le K, \tag{4}$$

(uniform Lipschitz condition) for all φ, $\psi \in B$ and $t \in [0,T]$, it follows that

$$|f(\varphi,t) - f(\psi,t)|^2 \vee |g(\varphi,t) - g(\psi,t)|^2 \le \bar{K}|\varphi - \psi|_B^2, \tag{5}$$

then the system (2) has a unique solution $X(t)$. Moreover, $X(t) \in \mathcal{M}^2((-\infty,T];R^d)$.

First, let us prove a useful lemma.

Lemma 3.1 Suppose that the weaken linear growth condition (4) holds. If $X(t)$ is the solution to the system (2), then

$$E\Big(\sup_{-\infty<t\le T}|X(t)|^2\Big) \le E|\xi|_B^2 + C\,\mathrm{e}^{6\bar{K}T(T+1)G(\beta)}, \tag{6}$$

where

$$C = 3N^2E|\xi|_B^2 + 6T(T+1)\Big(K + \frac{2K^2(\beta)\bar{K}}{1-2M^2(\beta)}E|\xi|_B^2\Big).$$

In addition, $X(t) \in \mathcal{M}^2((-\infty,T];R^d)$.

The proofs of Theorem 3.1 and Lemma 3.1 are similar to the proof of [2].

Define the Picard sequence

$$X^n(t) = \xi(0) + \int_0^t f(X_s^{n-1},\, s)\mathrm{d}s + \int_0^t g(X_s^{n-1},\, s)\mathrm{d}B(s), \tag{7}$$

it is easy to check that for any $n \ge 0$,

$$E\Big(\sup_{0\le s\le t}\big|X^{n+1}(s) - X^n(s)\big|^2\Big) \le \frac{C\,[J(\beta)t]^n}{n!}, \quad 0\le t\le T, \tag{8}$$

where

$$C = 4T(T+1)\big(K + \bar{K}E|\xi|_B^2\big), \quad J(\beta) = 2\bar{K}(T+1)G(\beta).$$

Then the estimate of error for the approximate solution and accurate solution is follows.

Theorem 3.2 Let $X(t)$ be the unique solution of the system (2), $X^n(t)$ be defined by (7). If the conditions (4) and (5) hold, then for any $n\ge 1$, it then follows

$$E\Big(\sup_{0\le t\le T}\big|X^n(t) - X(t)\big|^2\Big) \le \frac{2C\,[J(\beta)T]^n}{n!}\,\mathrm{e}^{2J(\beta)T}. \tag{9}$$

Proof One easily finds that

$$\begin{aligned}
E\Big(\sup_{0\le s\le t}\big|X^n(t) - X(t)\big|^2\Big) &\le J(\beta)\int_0^t E\Big(\sup_{0\le r\le s}\big|X^{n-1}(r) - X(r)\big|^2\Big)\mathrm{d}s \\
&\le 2J(\beta)\int_0^t E\Big(\sup_{0\le r\le s}\big|X^n(r) - X^{n-1}(r)\big|^2\Big)\mathrm{d}s \\
&\quad +2J(\beta)\int_0^t E\Big(\sup_{0\le r\le s}\big|X^n(r) - X(r)\big|^2\Big)\mathrm{d}s.
\end{aligned}$$

Substituting (8) into the above expression, then

$$E\Big(\sup_{0\le s\le t}\big|X^n(s) - X(s)\big|^2\Big) \le \frac{2C\,[J(\beta)T]^n}{n!} + 2J(\beta)\int_0^t E\Big(\sup_{0\le r\le s}\big|X^n(r) - X(r)\big|^2\Big)\mathrm{d}s.$$

Making use of the Gronwall inequality, as $0 \le t \le T$, we have

$$E\Big(\sup_{0\le s\le t}\big|X^n(s)-X(s)\big|^2\Big) \le \frac{2C\,[J(\beta)T]^n}{n!}\,\mathrm{e}^{2J(\beta)T},$$

as $t=T$, the assertion (9) is the required result. The proof is complete.

For stochastic functional differential equations with infinite delay

$$\mathrm{d}X(t)=f(X_t,t)\mathrm{d}t+g(X_t,t)\mathrm{d}B(t), \quad t\in[0,\infty), \tag{10}$$

where $f(\cdot,t)$ and $g(\cdot,t)$ are mapping from $B\times[0,\infty)$ to R^d and $R^{d\times m}$. If the existence-and-uniqueness theorem hold on every finite interval $[0,T]$, then the system (10) has a unique solution $X(t)$ on entire interval $(-\infty,\infty)$. It is called a global solution. In the similar way, Theorem 3.1 can be generalized as follows.

Theorem 3.3 Suppose that for each real number $T>0$ and each integer $n\ge 1$, there exists a positive constant $K_{T,n}$ such that for all $t\in[0,T]$ and all $\varphi,\psi\in B$ with $|\varphi|_B\vee|\psi|_B\le n$, it then follows that

$$\big|\,f(\varphi,t)-f(\psi,t)\,\big|^2\vee\big|\,g(\varphi,t)-g(\psi,t)\,\big|^2\le K_{T,n}|\varphi-\psi|_B^2.$$

Suppose further that for each $T>0$, there exists a positive constant K_T, such that for all $\varphi\in B$ and $t\in[0,T]$, it follows that

$$\big|\,f(\varphi,t)\,\big|^2\vee\big|\,g(\varphi,t)\,\big|^2\le K_T\big(\,1+|\varphi|_B^2\,\big).$$

Then the system (10) has a unique global solution $X(t)\in\mathcal{M}^2((-\infty,\infty);R^d)$.

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B 空间中无限时滞随机泛函微分方程解的存在唯一性

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摘 要: 本文研究抽象空间 B 中无限时滞随机泛函微分方程解的存在唯一性，在弱化的线性增长条件和一致 Lipschitz 条件下，得到无限时滞随机泛函微分方程在区间 $[0,\infty)$ 上存在唯一解，进而，得到近似解与精确解之间的误差估计。

关键词: 随机泛函微分方程；存在性；唯一性；无限时滞